

# HJORTH ANALYSIS OF GENERAL POLISH GROUP ACTIONS

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**ABSTRACT.** Hjorth has introduced a Scott analysis for general Polish group actions, and has asked whether his notion of rank satisfies a boundedness principle similar to the one of Scott rank - namely, the orbit equivalence relation is Borel if and only if Hjorth ranks are bounded.

We present the principles of Hjorth analysis and Hjorth rank, and answer Hjorth's question positively. From that we get a positive answer to a conjecture due to Hjorth - for every limit ordinal  $\alpha$ , the set of elements whose orbit is of complexity less than  $\alpha$  is a Borel set. We then show Nadel's theorem for Hjorth rank - the rank of  $x$  is no more than  $\omega_1^{ck(x)}$ .

## 1. INTRODUCTION

In 1965, Scott has introduced a method for completely characterizing a countable model by formulas of  $\mathcal{L}_{\omega_1, \omega}$ . This method has quickly opened the way to a better understanding of isomorphism of countable models. To name a few examples, it turned out that the isomorphism relation of models of a theory  $T$  is a well behaved and well understood "limit" of Borel equivalence relations, and that this isomorphism relation is Borel if and only if the ranks of the characterizing formulas of the  $T$  - models are uniformly bounded.

The isomorphism of countable models of a theory  $T$  is just another example of an orbit equivalence relation induced by a Polish action, namely the action of  $S_\infty$  on the collection of countable models,  $Mod_{\mathcal{L}}(T)$ . Hence, a natural question arises - can a similar method be developed for the general scenario? Since Scott analysis heavily uses the internal structure of the points of  $Mod_{\mathcal{L}}(T)$ , it was very unclear how the general method will look like.

A substantial progress was achieved by Hjorth in 2000, introducing a Scott analysis for actions of  $S_\infty$  [5]. The same idea was developed by him in [4, 6] to form a Scott analysis for general Polish group actions, but the work was never published.

In what follows, we review Hjorth's work and continue it, showing that Hjorth has indeed found a decent Scott analysis for general Polish group actions. We prove various interesting properties of the new Hjorth analysis.

Let us begin with a general description of the main results, which will not be complete without the following definition:

**Definition 1.1.** A *Scott analysis of Polish actions* is a method defining for each Polish  $G$  - space  $X$  a decreasing sequence of equivalence relations  $\equiv_\alpha$  and for each  $x \in X$  a countable ordinal  $\delta(x)$  such that:

- (1)  $\equiv_\alpha$  are Borel and invariant under  $G$ .
- (2)  $E_G^X = \bigcap_{\alpha < \omega_1} \equiv_\alpha$ .
- (3) The function  $\delta : X \rightarrow (\omega_1, <)$  is Borel and invariant under the action of  $G$ .
- (4) There is an  $\alpha < \omega_1$  such that for every  $x, y \in X$ ,  $x$  and  $y$  are orbit equivalent if and only if  $x \equiv_{\delta(x)+\alpha} y$ .

The first thing to be shown is that there is a Scott analysis of Polish actions. Sections 3 and 4, due to Hjorth, present the outlines of his method and show that it is indeed a Scott analysis of Polish actions. The construction relies on a relation  $\leq_\alpha$  which is non-symmetric and transitive. The relation is between pairs, each pair has an object of the Polish space and an open set of  $G$ . When we step up the ordinals,  $(x, U) \leq_\alpha (y, V)$  is trying to tell us more about how does the two actions, of  $U$  on  $x$  and of  $V$  on  $y$ , compare to each other. We then define the equivalence relation  $\equiv_\alpha$ . The definition is what we believe to be a somewhat improved version of the one originally given by Hjorth. Roughly speaking,  $x$  and  $y$  are  $\alpha$  equivalent if they belong to the same  $\Pi_\alpha^0$  invariant sets, although that is precisely true only for limit  $\alpha$ 's. It is then shown that  $\equiv_\alpha$  is Borel and invariant under  $G$ , and that the intersection of all  $\equiv_\alpha$  is the orbit equivalence relation. That leads to a different proof of the well known theorem of Sami - if all orbits are  $\Pi_\alpha^0$  then  $E_G^X$  is Borel.

Next, Hjorth rank  $\delta(x)$  is defined. The definition differs only cosmetically from the one originally given by Hjorth. It is shown that the rank is Borel definable and invariant under  $G$ . A back and forth argument proves Scott's isomorphism theorem for that scenario - if  $x \equiv_{\delta(x)+\omega} y$  then  $x$  and  $y$  are orbit equivalent.

The main result of section 5 is a boundedness principle for Hjorth rank:

**Theorem 1.2.** *Let  $(G, X)$  be a Polish action and  $\mathbb{B} \subset X$  an invariant Borel set. Then  $E_G^\mathbb{B}$  is Borel if and only if there is an  $\alpha$  such that for every  $x \in \mathbb{B}$ ,  $\delta(x) \leq \alpha$ .*

This is done by first showing that sets of the form  $U \cdot x$ , for  $U \subseteq G$  open, are Borel, and their complexity is almost the same as the complexity of the orbit  $G \cdot x$ . It is then proved that if for  $x \in X$ , the complexities of the  $U \cdot x$ 's are bounded, then Hjorth rank is no higher than their bound. A proof of the boundedness principle follows easily, as well as a Borel definition of

$$\{x : [x] \text{ is } \Pi_\alpha^0 \text{ for } \alpha < \beta\}$$

for  $\beta$  limit. That positively answers a conjecture of Hjorth (see [4]). The Becker Kechris theorem for the logic action is then reproved using the newly developed tools.

The last section is dedicated to the proofs of Nadel's and Sacks' theorems for Hjorth analysis:

**Theorem 1.3.** *(Nadel) For every  $x \in X$ ,  $\delta(x) \leq \omega_1^{ck(x)}$ .*

**Theorem 1.4.** *(Sacks) If for every  $x$ ,  $\delta(x) \leq \alpha$  or  $\delta(x) < \omega_1^{ck(x)}$ , then Hjorth ranks are bounded.*

The exact definition of  $\omega_1^{ck(x)}$  will be given later. Informally, this is the first ordinal not recursive in a sequence  $x_G$  that contains all the information about the action of  $G$  on  $x$ . The same techniques are then used to establish the complexities of Hjorth rank comparisons, such as

$$\{(x, y) : \delta(x) \leq \delta(y)\}$$

etc.

The next section intends to cover most of the knowledge this paper assumes, so that the text will be as self contained as possible. Although knowledge of forcing might help, it is not obligatory in any way - the reader can safely use the standard definition of Vaught transforms, and translate the single forcing proof of the paper to Vaught transforms' terms.

We remark that Hjorth analysis can be rephrased in terms of forcing - see the statement of lemma 3.12 for more details.

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## 2. PRELIMINARIES

**2.1. Descriptive Set Theory of Polish Group Actions.** For the basics of descriptive set theory and much more, the reader is encouraged to consult [7].

We review basic facts about the descriptive set theory of Polish group actions. All the following can be found in [3, 1].

A *Polish Topology* is a separable topology induced by a complete metric. A *Polish Space* is a topological space whose topology is Polish. A subspace of a Polish space is Polish if and only if it is  $G_\delta$ . The product of a countable collection of Polish spaces is Polish. In particular,  $\omega^\omega$  and  $2^\omega$ , with the product topology of the discrete topologies, are both Polish. Universality properties indicate a strong connection between these two spaces and all other Polish spaces.

A *Standard Borel Space* is a set  $X$  equipped with a  $\sigma$ -algebra  $S$  such that there is a Polish topology  $\tau$  on  $X$  whose Borel  $\sigma$ -algebra is  $S$ . Given a Polish space  $X$ , the *Effros Borel space* of  $X$ ,  $F(X)$ , is the set of closed sets of  $X$  with the  $\sigma$ -algebra generated by

$$\{F \in F(X) : F \cap U \neq \emptyset\}$$

for  $U \subseteq X$  open. This is a standard Borel space.

A *Polish Group* is a topological group whose topology is Polish. One important example is  $S_\infty$ , the group of permutations of natural numbers. This group is readily contained in  $\omega^\omega$ , and one can easily show it is a  $G_\delta$  subset of  $\omega^\omega$ , and hence Polish.

Let  $G$  be a Polish group, and  $H \leq G$  a subgroup. Then  $H$  is Polish if and only if it is closed. In this case, the quotient  $G/H$ , as the set of left cosets of  $H$ , is a Polish space under the quotient topology. If  $H$  is normal, it is a Polish group as well.

A continuous action of a Polish group  $G$  on a Polish space  $X$  is called a *Polish action*. We will say that  $X$  is a *Polish  $G$ -space*. A Polish action naturally induces an orbit equivalence relation on  $X$ , which we will denote by  $E_G^X$ . For  $x \in X$ , the stabilizer of  $x$  is  $G_x = \{g : g \cdot x = x\}$ . Its left translations  $g \cdot G_x$  are all of the form  $G_{x,y} = \{g : g \cdot x = y\}$  for some  $y \in X$ . Since  $G_x$  is a closed subgroup,  $G/G_x$  is a Polish space. Consider the canonical bijection

$$G/G_x \rightarrow G \cdot x$$

It is clearly continuous, so that  $G \cdot x$  is the continuous injective image of a Polish space, hence Borel. However, the canonical bijection is a homeomorphism only when  $G \cdot x$  is Polish:

**Theorem 2.1. (Effros)**  $G/G_x \rightarrow G \cdot x$  is an homeomorphism iff  $G \cdot x$  is  $G_\delta$  iff  $G \cdot x$  is non-meager in its topology.

An immediate corollary is that whenever an orbit is non meager, it must be  $G_\delta$ .

The orbit equivalence relation  $E_G^X$  is analytic, but not always Borel. If it is Borel, there is an  $\alpha < \omega_1$  such that all orbits are  $\Pi_\alpha^0$ . It turns out that the opposite is true as well:

**Theorem 2.2.** (*Sami*) Let  $X$  be a Polish  $G$  - space. If there is an  $\alpha < \omega_1$  such that all orbits are  $\Pi_\alpha^0$  then  $E_G^X$  is Borel.

*Proof.* Define for  $\alpha$  countable the following equivalence relation  $E_\alpha$  :

$$xE_\alpha y \iff \forall A \text{ } \Pi_\alpha^0 \text{ invariant : } x \in A \iff y \in A$$

There is a universal set for  $\Pi_\alpha^0$  invariant sets - namely, there is  $U \subseteq \omega^\omega \times X$  a  $\Pi_\alpha^0$  set such that the set of sections  $\{U_f : f \in \omega^\omega\}$  is the set of  $\Pi_\alpha^0$  invariant sets of  $X$ . Now it is easy to see that  $E_\alpha$  is  $\Pi_1^1$ , and under the conditions of the theorem  $E_\alpha = E_G^X$  for a large enough  $\alpha$ .

In fact, using Louveau's separation theorem of [9], one can show that the  $E_\alpha$ 's are Borel, under the additional assumption that  $X$  is recursively presented.  $\square$

We mention another characterization of Borel orbit equivalence relations, due to Becker and Kechris:

**Theorem 2.3.** Let  $X$  be a Polish  $G$  - space. The following are equivalent:

- (1)  $E_G^X$  is Borel.
- (2) The map  $x \rightarrow G_x$  from  $X$  into  $F(G)$  is Borel.
- (3) The map  $(x, y) \rightarrow G_{(x, y)}$  from  $X^2$  into  $F(G)$  is Borel.

**2.2. The Logic Action.** One of the most important example of Polish actions is the *logic action*. Let  $\mathcal{L}$  be a countable relational language,  $\mathcal{L} = (R_i)_{i \in \omega}$  when  $R_i$  is an  $n_i$  - ary relation. We denote by  $Mod(\mathcal{L})$  the collection of countable models, and assume that all the models have the set of natural numbers as their universe. Then every  $\mathcal{M} \in Mod(\mathcal{L})$  can in fact be coded as an element of  $\Pi_{i \in \omega} 2^{\omega^{n_i}}$  (which is homeomorphic to  $2^\omega$ ). In particular,  $Mod(\mathcal{L})$  inherits the topology of  $\Pi_{i \in \omega} 2^{\omega^{n_i}}$ . This is exactly the topology generated by

$$A_{\phi, \bar{a}} = \{\mathcal{M} : \mathcal{M} \models \phi(\bar{a})\}$$

where  $\phi$  is an atomic formula or a negation of one, and  $\bar{a} \in \omega^{<\omega}$ .

This is indeed a very natural topology for  $Mod(\mathcal{L})$ , as the following theorem demonstrates:

**Theorem 2.4.** (*Lopez - Escobar*)  $B \subseteq Mod(\mathcal{L})$  is Borel invariant if and only if there is a sentence  $\phi \in \mathcal{L}_{\omega_1, \omega}$  such that  $B = Mod(\phi)$ .

A proof can be found in [3].

We now define a Polish action of  $S_\infty$  on  $Mod(L)$  by:

$$R^{gM}(a_1, \dots, a_n) \iff R^M(g^{-1}(a_1), \dots, g^{-1}(a_n))$$

This action is called the logic action, and it is easy to verify that the orbit equivalence relation is exactly the isomorphism of  $L$  - models.

In many cases in mathematics, we will want to study the isomorphism classes of models of a certain first order theory, for example, the isomorphism classes of groups or rings. We will therefore want to restrict the action of  $S_\infty$  only to the models of this theory. So let  $T$  be a first order theory in a countable language  $\mathcal{L}$ . The collection of countable models of  $T$ ,  $Mod_{\mathcal{L}}(T)$ , is a Borel invariant subset of  $Mod(\mathcal{L})$ . The continuous action of  $S_\infty$  on the Borel invariant subset  $Mod_{\mathcal{L}}(T)$  induces the isomorphism of models of  $T$  as its orbit equivalence relation. That important example is one of the reasons we will prefer to state our theorem for  $B$  Borel invariant and not necessarily Polish.

**2.3. Scott Analysis.** We review the basic properties of Scott analysis, as were established in [15]. A more detailed review can also be found in [11, 3].

**Definition 2.5.** Let  $\mathcal{M}, \mathcal{N} \in \text{Mod}(\mathcal{L})$ ,  $\bar{a}, \bar{b} \in \omega^{<\omega}$  of the same length.

- $(\mathcal{M}, \bar{a}) \equiv_0 (\mathcal{N}, \bar{b})$  if for every  $\phi(\bar{x})$  atomic,  $\mathcal{M} \models \phi(\bar{a}) \iff \mathcal{N} \models \phi(\bar{b})$ .
- $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{N}, \bar{b})$  if for every  $c \in \omega$  there is  $d \in \omega$  s.t.  $(\mathcal{M}, \bar{a} \frown c) \equiv_\alpha (\mathcal{N}, \bar{b} \frown d)$  and for every  $d \in \omega$  there is  $c \in \omega$  s.t.  $(\mathcal{N}, \bar{b} \frown d) \equiv_\alpha (\mathcal{M}, \bar{a} \frown c)$ .
- For  $\lambda$  limit,  $(\mathcal{M}, \bar{a}) \equiv_\lambda (\mathcal{N}, \bar{b})$  if for every  $\alpha < \lambda$ ,  $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{N}, \bar{b})$ .

Saying that  $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{N}, \bar{b})$  expresses a certain similarity between the tuple  $\bar{a}$  in  $\mathcal{M}$  and the tuple  $\bar{b}$  in  $\mathcal{N}$ . The similarity improves as  $\alpha$  increases. We will see that if  $\alpha$  is large enough,  $(\mathcal{M}, \bar{a}) \simeq (\mathcal{N}, \bar{b})$ , which is, there is an isomorphism between  $\mathcal{M}$  and  $\mathcal{N}$  which takes  $\bar{a}$  to  $\bar{b}$ .

**Definition 2.6.** We say that  $\mathcal{M} \equiv_\alpha \mathcal{N}$  if  $(\mathcal{M}, \emptyset) \equiv_\alpha (\mathcal{N}, \emptyset)$ .

$\equiv_\alpha$  is a decreasing sequence of equivalence relations. The following is easily proved by induction:

**Proposition 2.7.** For every  $\alpha$ ,  $\equiv_\alpha$  is Borel, and in fact  $\Pi_{1+2\alpha}^0$ .

This equivalence relation carries information about the collection of  $\Pi_\alpha^0$  invariant sets to which the models belong, as the following theorem demonstrates:

**Theorem 2.8.** Let  $A \subseteq \text{Mod}(\mathcal{L})$  be a  $\Pi_\alpha^0$  invariant set. If  $\mathcal{M} \equiv_{\omega \cdot \alpha} \mathcal{N}$  then  $\mathcal{M} \in A \iff \mathcal{N} \in A$ .

Toward defining Scott rank, we need to show:

**Proposition 2.9.** Given  $\mathcal{M} \in \text{Mod}(\mathcal{L})$ , there is  $\alpha < \omega_1$  such that if  $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{M}, \bar{b})$  then  $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{M}, \bar{b})$ . Moreover, for such an  $\alpha$ ,  $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{M}, \bar{b})$  implies that for all  $\beta$ ,  $(\mathcal{M}, \bar{a}) \equiv_\beta (\mathcal{M}, \bar{b})$ .

*Proof.* Define for every  $\alpha < \omega_1$  :  $A_\alpha = \{(\bar{a}, \bar{b}) : \neg(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{M}, \bar{b})\}$ . This is an increasing sequence of subsets of  $\omega^{<\omega}$ , and strictly increasing till it stabilizes. Hence, it must stabilize at a certain point.  $\square$

**Definition 2.10.** For  $\mathcal{M} \in \text{Mod}(\mathcal{L})$ ,  $\delta(\mathcal{M})$ , the *Scott rank* of  $\mathcal{M}$ , is the least  $\alpha$  such that for all  $\bar{a}, \bar{b} \in \omega^{<\omega}$ ,  $(\mathcal{M}, \bar{a}) \equiv_\alpha (\mathcal{M}, \bar{b})$  implies  $(\mathcal{M}, \bar{a}) \equiv_{\alpha+1} (\mathcal{M}, \bar{b})$ .

If  $\mathcal{M} \equiv_{\delta(\mathcal{M})+\omega} \mathcal{N}$  then  $(\mathcal{M}, \bar{a}) \equiv_{\delta(\mathcal{M})} (\mathcal{N}, \bar{b})$  implies  $(\mathcal{M}, \bar{a}) \equiv_{\delta(\mathcal{M})+1} (\mathcal{N}, \bar{b})$ . The main theorem follows:

**Theorem 2.11. (Scott Isomorphism Theorem)** Let  $\mathcal{M}, \mathcal{N} \in \text{Mod}(\mathcal{L})$  such that  $\mathcal{M} \equiv_{\delta(\mathcal{M})+\omega} \mathcal{N}$ . Then  $\mathcal{M} \simeq \mathcal{N}$ .

*Proof.* This is done by a back & forth argument.  $\square$

For a first order theory  $T$  in a countable language  $\mathcal{L}$ , we denote by  $\simeq_T$  the isomorphism of models of  $T$ . We can now state 3 characterizations of Borel  $\simeq_T$ :

**Theorem 2.12. (Becker - Kechris)**  $\simeq_T$  is Borel if and only if there is an  $\alpha < \omega_1$  such that for every  $\mathcal{M} \in \text{Mod}_{\mathcal{L}}(T)$ ,  $\delta(\mathcal{M}) < \alpha$

**Theorem 2.13. (Becker - Kechris)** The following are equivalent:

- (1)  $\simeq_T$  is Borel

- (2) The set  $\{ (\mathcal{M}, \mathcal{N}, \bar{a}, \bar{b}) : \mathcal{M}, \mathcal{N} \in \text{Mod}_{\mathcal{L}}(T); \bar{a}, \bar{b} \in \omega^{<\omega} \text{ of the same length}; (\mathcal{M}, \bar{a}) \simeq \mathcal{N}, \bar{b}) \}$  is Borel.
- (3) The set  $\{ (\mathcal{M}, \bar{a}, \bar{b}) : \mathcal{M} \in \text{Mod}_{\mathcal{L}}(T); \bar{a}, \bar{b} \in \omega^{<\omega} \text{ of the same length}; (\mathcal{M}, \bar{a}) \simeq \mathcal{M}, \bar{b}) \}$  is Borel.

A careful analysis of the lightface complexity of  $\equiv_{\alpha}$  has opened way to the following result, due to Nadel [12]:

**Theorem 2.14.** For every  $\mathcal{M}$ ,  $\delta(\mathcal{M}) \leq \omega_1^{ck(\mathcal{M})}$ .

We recall that for  $x \in 2^{\omega}$ ,  $\omega_1^{ck(x)}$  is the first ordinal not computable from  $x$ . Any countable model  $\mathcal{M}$  can be identified with an element of  $2^{\omega}$ , as explained above.

Sacks has shown that if for all  $\mathcal{M}$ ,  $\delta(\mathcal{M}) < \omega_1^{ck(\mathcal{M})}$ , the isomorphism relation is Borel:

**Theorem 2.15.**  $(\text{Sacks}) \simeq$  is Borel if and only if there is  $\alpha < \omega_1$  such that for all  $\mathcal{M}$ ,  $\delta(\mathcal{M}) \leq \alpha$  or  $\delta(\mathcal{M}) < \omega_1^{ck(\mathcal{M})}$ .

Proofs of all of the above theorems can be found in [3].

**2.4. Vaught Transforms and Forcing.** Let  $\langle \mathbb{P}, \leq \rangle$  be a partial order. If  $p \leq q$  we say that  $p$  extends  $q$ , and if  $p, q$  have a common extension they are *compatible*. A set  $D \subseteq \mathbb{P}$  is *dense* if every  $p \in \mathbb{P}$  has an extension in  $D$ . For  $\mathbb{V}$  a model of  $ZFC$ , we say that  $G \subseteq \mathbb{P}$  is a *generic filter over  $\mathbb{V}$*  if:  $p \in G$  and  $q \geq p$  implies  $q \in G$ , every pair  $p, q \in G$  has a common extension in  $G$ , and for every  $D \subseteq \mathbb{P}$  dense,  $G \cap D \neq \emptyset$ . The generic extension of  $\mathbb{V}$  by  $G$  is the minimal model of  $ZFC$  containing both  $\mathbb{V}$  and  $G$ , and is denoted by  $\mathbb{V}[G]$ . For names  $\tau_1, \dots, \tau_n$ , a formula  $\phi$  and  $p \in \mathbb{P}$ ,  $p \Vdash \phi(\tau_1, \dots, \tau_n)$  if for every generic filter  $G$  such that  $p \in G$ ,  $\mathbb{V}[G] \models \phi(\tau_1[G], \dots, \tau_n[G])$ .

Given a Polish space  $X$ , let  $\mathbb{P}_I$  be the partial order of non meager Borel subsets of  $X$  ordered by inclusion (in fact,  $\mathbb{P}_I$  is a partial order of codes of non meager Borel sets). For  $G$  a generic filter in  $\mathbb{P}_I$ , there is a unique  $x \in X$  in  $\mathbb{V}[G]$  such that

$$G = \{B \subseteq X : B \in \mathbb{P}_I; x \in B\}$$

so that in fact  $\mathbb{V}[G] = \mathbb{V}[x]$ . Let  $x^*$  be a canonical name for this generic element. Then for  $C$  a non meager Borel set:

$$C \Vdash x^* \in \check{B} \iff C - B \text{ is meager.}$$

In fact,  $\mathbb{P}_I$  is equivalent to Cohen forcing over  $X$ , which is, forcing with the nonempty open subsets of  $X$ , since it is densely embedded in the separative quotient of  $\mathbb{P}_I$ .

**Definition 2.16.** (Vaught Transforms) Let  $X$  be a Polish  $G$ -space,  $A \subseteq X$  and  $U \subseteq G$  open.

$$A^{*U} = \{x : \{g \in U : g \cdot x \in A\} \text{ is comeager in } U\}$$

$$A^{\Delta U} = \{x : \{g \in U : g \cdot x \in A\} \text{ is non meager in } U\}$$

We write  $A^*$  and  $A^{\Delta}$  for  $A^{*G}$  and  $A^{\Delta G}$ , respectively.

Using the above, it is easy to show that  $x \in A^{*U} \iff U \Vdash_{\mathbb{P}_I} g^* \cdot \check{x} \in \check{A}$ , while  $\mathbb{P}_I$  are the non meager Borel subsets of  $G$ , and  $g^*$  is the name of the generic element.

We summarize a few important properties of Vaught transforms:

**Lemma 2.17.** Let  $X$  be a Polish  $G$ -space.  $A, A_n \subseteq X$ ,  $U \subseteq G$  open,  $U_n$  a basis for the topology of  $G$ .

- (1)  $A^{\Delta}$  and  $A^*$  are invariant, and  $A$  is invariant iff  $A = A^{\Delta}$  iff  $A = A^*$ .

- (2)  $A^{\Delta U} = X - (X - A)^{*U}$ .
- (3) If  $A = \bigcup A_n$  then  $A^{\Delta U} = \bigcup A_n^{\Delta U}$ . If  $A = \bigcap A_n$  then  $A^{*U} = \bigcap A_n^{*U}$ .
- (4) If  $A$  is  $\Pi_\alpha^0$  then  $A^{*U}$  is  $\Pi_\alpha^0$ . If  $A$  is  $\Sigma_\alpha^0$  then  $A^{\Delta U}$  is  $\Sigma_\alpha^0$ .
- (5)  $A^{*U} = \bigcap \{A^{\Delta U_n} : U_n \subseteq U\}$ .

**2.5. Better Topologies.** Refinement of Polish topologies is a very common tool in descriptive set theory. Given a Polish space  $(X, \tau)$  and a sequence  $B_n$  of Borel sets, there is a Polish topology refining  $\tau$  such that for all  $n$ ,  $B_n$  is clopen. When a Polish  $G$  - space  $X$  is given, refining the topology while maintaining the continuity of the action is a harder problem. Results of Becker, Kechris and Hjorth have led to the following:

**Theorem 2.18.** *Let  $(X, \tau)$  be a Polish  $G$  - space, and  $U \subseteq G$  a countable collection of open sets of  $G$ .  $\mathcal{A}$  a countable collection of  $\Sigma_\alpha^0(\mathbf{X}, \tau)$ . There is a Polish  $\tau \subseteq \sigma \subseteq \Sigma_\alpha^0(\mathbf{X}, \tau)$  s.t.  $(X, \sigma)$  is a Polish  $G$  - action and  $\mathcal{A}^{\Delta U} \subseteq \sigma$ .*

In particular, if  $X$  is a Polish  $G$  - space, and  $B \subseteq X$  is Borel invariant, then there is a Polish topology on  $B$  such that  $B$  is a Polish  $G$  - space. Proofs of the above can be found in [3, 1].

### 3. HJORTH ANALYSIS

Let  $(G, X)$  be a general Polish action, and  $\mathfrak{B}_0$  a countable basis for the topology of  $G$ . Our main task in the following chapter is defining the equivalence relations  $\equiv_\alpha$  which will approximate  $E_G^X$ , as explained in the introduction. We first define a reflexive, transitive and non-symmetric relation between pairs of an element of  $x$  and an open subset of  $G$ :

**Definition 3.1.** For  $V_0, V_1 \subseteq G$  open and non-empty, and  $x_0, x_1 \in X$  :

$$(x_0, V_0) \leq_1 (x_1, V_1)$$

if

$$\overline{V_0 \cdot x_0} \subseteq \overline{V_1 \cdot x_1}.$$

At successor stages :

$$(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$$

if for all  $W_0 \subseteq V_0$  open and non-empty, there is  $W_1 \subseteq V_1$  open and non-empty such that:

$$(x_1, W_1) \leq_\alpha (x_0, W_0).$$

For  $\lambda$  a limit:

$$(x_0, V_0) \leq_\lambda (x_1, V_1)$$

if for every  $\alpha < \lambda$

$$(x_0, V_0) \leq_\alpha (x_1, V_1).$$

**Lemma 3.2.** *For  $W_0 \subseteq V_0$ ,  $W_1 \supseteq V_1$ , all open and non-empty,*

$$(x_0, V_0) \leq_\alpha (x_1, V_1)$$

*implies*

$$(x_0, W_0) \leq_\alpha (x_1, W_1).$$

*Proof.* Trivial. □

The next lemma will be needed when proving the Borel definability of  $\leq_\alpha$ :

**Lemma 3.3.**  $(x_0, V_0) \leq_{\alpha+1} (x_1, V_1)$  iff for all  $W_0 \subseteq V_0$  in  $\mathfrak{B}_0$  there is  $W_1 \subseteq V_1$  in  $\mathfrak{B}_0$  with  $(x_1, W_1) \leq_\alpha (x_0, W_0)$ .

*Proof.* Follows from lemma 3.2. □

**Lemma 3.4.** (a) Each  $\leq_\alpha$  is transitive.

(b) For  $\alpha < \beta$ , if  $(x_0, W_0) \leq_\beta (x_1, W_1)$  then  $(x_0, W_0) \leq_\alpha (x_1, W_1)$ .

*Proof.* (a) Immediate.

(b) The cases  $\beta = 1$  and  $\beta$  limit are obvious. We divide the case  $\beta = \gamma + 1$  into 3 subcases:  $\gamma = 1$ ,  $\gamma$  is a successor and  $\gamma$  is a limit.

$\gamma = 1$ : Assume  $(x_0, W_0) \leq_2 (x_1, W_1)$  and  $\overline{W_0 \cdot x_0} \not\subseteq \overline{W_1 \cdot x_1}$ . So there is an open set  $O$  in  $X$  that intersects  $\overline{W_0 \cdot x_0}$  but doesn't intersect  $\overline{W_1 \cdot x_1}$ . By the continuity of the action, we can find  $U_0 \subseteq W_0$  such that  $\overline{U_0 \cdot x_0} \subseteq O$ . Which leads to a contradiction.

The other 2 subcases involve only standard induction arguments. □

We are now ready to define the equivalence relation:

**Definition 3.5.** Let  $x_0, x_1$  in  $X$ . We will say that  $x_0 \equiv_\alpha x_1$  iff for  $i \in \{0, 1\}$  and for  $V_i \subseteq G$  open and nonempty, there is  $V_{1-i} \subseteq G$  open and nonempty such that  $(x_{1-i}, V_{1-i}) \leq_\alpha (x_i, V_i)$ .

**Lemma 3.6.**  $\equiv_\alpha$  is an equivalence relation.

**Lemma 3.7.**  $x_0 \equiv_\alpha x_1$  iff for  $i \in \{0, 1\}$  and for  $V_i \in \mathcal{B}_0$ , there is  $V_{1-i} \in \mathcal{B}_0$  such that  $(x_{1-i}, V_{1-i}) \leq_\alpha (x_i, V_i)$ .

**Lemma 3.8.** For every  $\alpha$  and every  $g \in G$ :  $(x_0, V_0) \leq_\alpha (g \cdot x_0, V_0 \cdot g^{-1})$ .

*Proof.* By transfinite induction on  $\alpha$ . The cases  $\alpha = 1$  and  $\alpha$  limit are trivial. For a successor  $\alpha$ , given  $W_0 \subseteq V_0$ , by the induction hypothesis we have  $(g \cdot x_0, W_0 \cdot g^{-1}) \leq_\alpha (x_0, W_0)$ , so  $W_0 \cdot g^{-1}$  is the open set we're looking for. □

**Corollary 3.9.**  $\equiv_\alpha$  is invariant under the action of  $g$ .

*Proof.* We show that  $x_0 \equiv_\alpha g \cdot x_0$ , using Lemma 3.8. □

The next task is showing that  $\equiv_\alpha$  is Borel:

**Proposition 3.10.** Let  $V_n$  be an enumeration of  $\mathfrak{B}_0$ . Then for all  $\alpha < \omega_1$

$$\mathfrak{R}_\alpha = \{(x_0, x_1, n, m) : (x_0, V_m) \leq_\alpha (x_1, V_n)\}$$

is a  $\Pi_{\alpha+k(\alpha)}^0$  set for some  $k(\alpha) \in \omega$  (and  $\Pi_\alpha^0$  for  $\alpha$  limit).

*Proof.* By induction on  $\alpha$ . For the case  $\alpha = 1$ , notice that for fixed  $n, m$ , the set of  $x_0, x_1$  such that  $(x_0, x_1, n, m) \in \mathfrak{R}_1$  is  $G_\delta$ . For  $\alpha$  successor, use lemma 3.3. □



**Theorem 3.11.** *The equivalence relation  $x_0 \equiv_\alpha x_1$  is  $\Pi_{\alpha+\mathbf{m}(\alpha)}^0$  (while  $m(\alpha) = 2$  for  $\alpha$  limit) .*

*Proof.* Follows from lemma 3.7 and the previous proposition. □

Hjorth has shown in [6] that  $\equiv_\alpha$  is potentially  $\Pi_{\alpha+1}^0$ .

The relations  $\leq_\alpha$  can be defined in terms of Vaught transforms or forcing:

**Proposition 3.12.**  *$(x, U) \leq_\alpha (y, W)$  if and only if for every  $A$  a  $\Pi_\alpha^0$  set, if  $y \in A^{*W}$  then  $x \in A^{*U}$ .*

*Restating in terms of forcing:  $(x, U) \leq_\alpha (y, W)$  if and only if for every  $A$  a  $\Pi_\alpha^0$  set, if  $W \Vdash g^*y \in A$  then  $U \Vdash g^*x \in A$  ( $g^*$  is the canonical name of the generic element ).*

*Proof.* By induction over  $\alpha$ .

$\alpha = 1 : (\Rightarrow)$  Assume  $U \cdot x \subseteq \overline{W \cdot y}$ . Let  $A$  be closed, and  $W \Vdash g^*y \in A$ , which is:

$$W \Vdash g^* \in \{g : g \cdot y \in A\}$$

This can happen only if  $W - \{g : g \cdot y \in A\}$  is meager, or equivalently in this case, empty. We can now deduce that  $W \cdot y \subseteq A$  so  $U \cdot x \subseteq \overline{W \cdot y} \subseteq A$ . Repeating the same argument we get  $U \Vdash g^*x \in A$ .

$(\Leftarrow)$  Let  $A = \overline{W \cdot y}$ . Then  $W \Vdash g^*y \in A$ , and hence  $U \Vdash g^*x \in A$ . This can happen only if  $U \subseteq \{g : g \cdot x \in A\}$ , so  $U \cdot x \subseteq A$ , as wanted.

Assume for  $\beta < \alpha$ , we prove for  $\alpha$ :  $(\Rightarrow)$  Assume  $(x, U) \leq_\alpha (y, W)$ . Let  $A = \bigcap_{n \in \omega} B_n$  for  $B_n \Sigma_{\beta_n}^0$  sets,  $\beta_n < \alpha$ . Assume  $W \Vdash g^*y \in A$ , and assume by way of contradiction that  $U \nVdash g^*x \in A$ . Thus, there is  $U' \subseteq U$  and  $n \in \omega$  such that  $U' \Vdash g^*x \notin B_n$ . There is  $W' \subseteq W$  such that  $(y, W') \leq_{\beta_n} (x, U')$ , and so, using the induction hypothesis,  $W' \Vdash g^*y \notin B_n$ . Contradiction.

$(\Leftarrow)$  Assume  $(x, U) \not\leq_{\alpha+1} (y, W)$  ( the limit case is trivial ). There is  $U' \subseteq U$  in  $\mathcal{B}_0$  such that for every  $W' \subseteq W$  in  $\mathcal{B}_0$ ,  $(y, W') \not\leq_\alpha (x, U')$ . Using the induction hypothesis, there is  $B_{W'}$  a  $\Pi_\alpha^0$  set such that  $U' \Vdash g^*x \in B_{W'}$  but  $W' \nVdash g^*y \in B_{W'}$ . We then find  $W'' \subseteq W'$  such that  $W'' \Vdash g^*y \notin B_{W'}$ . We denote by  $A$  the set

$$\bigcup_{W' \subseteq W; W' \in \mathcal{B}_0} (X - B_{W'})$$

which is  $\Pi_{\alpha+1}^0$ . The above means that  $W \Vdash g^*y \in A$ . However,  $U' \Vdash g^*x \notin A$ , so  $U \nVdash g^*x \in A$ . As wanted. □

*Remark 3.13.* Using this equivalent definition,  $(x, B) \leq_\alpha (y, C)$  has a meaning for any  $B, C$  Borel and non-meager. At the same time, such an expansion of the definition is redundant.

**Theorem 3.14.** *If  $x \equiv_\alpha y$  then  $x$  and  $y$  belong to the same  $\Pi_\alpha^0$  invariant sets.*

*Proof.* Assume  $x \in A$  for  $A$  a  $\Pi_\alpha^0$  invariant set. As  $A$  is invariant,  $x \in A^{*G}$ . Since  $x \equiv_\alpha y$ , there is a non empty and open  $W$  such that  $(y, W) \leq_\alpha (x, G)$ . The previous lemma then implies  $y \in A^{*W}$ . In particular, there is a  $g$  such that  $g \cdot y \in A$ . By the invariance of  $A$ ,  $y$  must be in  $A$ . □

That leads us to the more elegant definition of  $\equiv_\alpha$  for limit  $\alpha$ 's:

**Corollary 3.15.** *For limit  $\alpha$ ,  $x \equiv_\alpha y$  if and only if  $x$  and  $y$  belong to the same invariant sets.*

*Proof.* The previous theorem and theorem 3.11. □

**Theorem 3.16.** (*Sami's Theorem*) Let  $B \subseteq X$  be an invariant Borel set, and assume that for every  $x \in B$ ,  $[x]_G$  is  $\Pi_\alpha^0$ . Then  $E_G^B$  is Borel, and in fact it is  $\Pi_{\alpha+m(\alpha)}^0 \cap (B \times B)$  for some  $m(\alpha) \in \omega$ .

*Proof.* The above theorem shows that in this case,  $\equiv_\alpha$  coincides with the orbit equivalence relation on  $B$ .  $\square$

The same argument proves the following:

**Corollary 3.17.** Let  $B \subseteq X$  be an invariant Borel set, and assume the orbit equivalence relation  $E_G^B$  is Borel. Then there is an  $\alpha < \omega_1$  such that  $E_G^B = \equiv_\alpha \cap (B \times B)$ .

#### 4. HJORTH RANK

We define now the Hjorth rank of an element of  $X$ . A careful definition is required, since we want it to be both Borel definable and invariant under  $G$ . In Scott analysis, the rank of  $x$  was the first place in which we can step up “for free”. Here, stepping up will have an infinitesimal price of shrinking and expanding the open sets involved. Also, stepping up will only be allowed when the open sets are in  $\mathfrak{B}_0$ .

**Definition 4.1.** For  $x \in X$ , let  $\delta(x)$  be the least  $\alpha$  such that for every  $V_0, V_1, W_0, W_1$  in  $\mathfrak{B}_0$ , if

$$\overline{W_0} \subseteq V_0 \text{ \& } \overline{V_1} \subseteq W_1 \text{ \& } (x, V_0) \leq_\alpha (x, V_1)$$

then  $(x, W_0) \leq_{\alpha+1} (x, W_1)$ .

We claim that such a  $\delta(x)$  exists, or equivalently, that there is an ordinal with the above properties. In fact, given an  $x \in X$ ,  $U_1, U_2$  in  $\mathfrak{B}_0$ , either there is  $\beta_{u_1, u_2} < \omega_1$  such that  $(x, U_1) \leq_{\beta_{u_1, u_2}} (x, U_2)$  and  $(x, U_1) \not\leq_{\beta_{u_1, u_2}+1} (x, U_2)$ , or  $(x, U_1) \leq_\alpha (x, U_2)$  for every  $\alpha < \omega_1$ . Now take  $\gamma(x) = (\sup_{U_1, U_2 \in \mathfrak{B}_0} \beta_{u_1, u_2}) + 1$ .  $\gamma(x)$  has the above property.

We'll call  $\delta(x)$  the *Hjorth rank* of  $x$ .

**Lemma 4.2.** We may assume that the basis  $\mathfrak{B}_0$  has the following property:

- For every  $\overline{U} \subseteq W$  in  $\mathfrak{B}_0$  and every  $g \in G$ , there are  $V_1, V_2$  in  $\mathfrak{B}_0$  such that  $U \cdot g \subseteq V_1 \subseteq V_2 \subseteq W \cdot g$  and  $\overline{V_1} \subseteq V_2$ .

*Proof.* Fix a right invariant metric  $d$  (not necessarily complete), and let  $\mathfrak{B}_0$  be the set of balls of rational radii around a dense set.  $\square$

**Lemma 4.3.**  $xE_G^X y$  implies  $\delta(x) = \delta(y)$ .

*Proof.* Let  $\alpha$  be the rank of  $x$ . Assume  $(g \cdot x, V) \leq_\alpha (g \cdot x, U)$ ,  $\overline{O} \subseteq V$  and  $\overline{U} \subseteq W$ . Equivalently,  $(x, V \cdot g) \leq_\alpha (x, U \cdot g)$ ,  $\overline{O \cdot g} \subseteq V \cdot g$  and  $\overline{U \cdot g} \subseteq W \cdot g$ . Using the previous Lemma, there are  $V_1, V_2, O_1, O_2$  in  $\mathfrak{B}_0$  such that

$$U \cdot g \subseteq V_1 \subseteq V_2 \subseteq W \cdot g$$

$$\overline{V_1} \subseteq V_2$$

$$O \cdot g \subseteq O_1 \subseteq O_2 \subseteq V \cdot g$$

$$\overline{O_1} \subseteq O_2.$$

Trivially,  $(x, V \cdot g) \leq_\alpha (x, U \cdot g)$  implies  $(x, O_2) \leq_\alpha (x, V_1)$ . Since  $\alpha$  is the rank of  $x$ , we can step up to  $(x, O_1) \leq_{\alpha+1} (x, V_2)$ . This last one trivially implies  $(x, O \cdot g) \leq_{\alpha+1} (x, W \cdot g)$ , or equivalently,  $(g \cdot x, O) \leq_{\alpha+1} (g \cdot x, W)$ .  $\square$

We have only allowed stepping up when the open sets are in  $\mathcal{B}_0$ . However, this restriction does not bother us too much:

**Lemma 4.4.** *Suppose  $\delta(x) = \delta$ ,  $V_0, V_1$  open and nonempty, and  $(x, V_0) \leq_{\delta+1} (x, V_1)$ . Then  $(x, V_0) \leq_{\delta+2} (x, V_1)$ , and in fact, for every  $\alpha$ ,  $(x, V_0) \leq_\alpha (x, V_1)$ .*

*Proof.* Let  $W_0 \subseteq V_0$  in  $\mathcal{B}_0$ . We choose  $\overline{W'_0} \subseteq W_0$  in  $\mathcal{B}_0$ . There is  $W'_1 \subseteq V_1$  in  $\mathcal{B}_0$  such that  $(x, W'_1) \leq_\delta (x, W'_0)$ . We choose  $\overline{W_1} \subseteq W'_1$  in  $\mathcal{B}_0$  and step up to

$$(x, W_1) \leq_{\delta+1} (x, W_0).$$

The furthermore part is proved by induction on  $\alpha \geq \delta + 1$ .  $\square$

*Remark 4.5.* The definition of  $\delta(x)$  depends on the choice of the basis  $\mathcal{B}_0$ . However, the previous lemma shows that for a given  $x \in X$ , a different choice of basis changes the rank by at most 1.

**Lemma 4.6.** *If  $x_0 \equiv_{\delta+1} x_1$  for  $\delta \geq \delta(x_1)$  then*

$$(x_0, V_0) \leq_{\delta+1} (x_1, V_1)$$

*implies  $(x_0, V_0) \leq_{\delta+2} (x_1, V_1)$ .*

*Proof.* Let  $V'_0 \subseteq V_0$ . Let  $V_1^* \subseteq V_1$  be such that  $(x_1, V_1^*) \leq_{\delta+1} (x_0, V'_0)$ . By transitivity of  $\leq$ ,  $(x_1, V_1^*) \leq_{\delta+1} (x_1, V_1)$ , hence  $(x_1, V_1^*) \leq_{\delta+2} (x_1, V_1)$ . We can then find  $V'_1 \subseteq V_1$  such that  $(x_1, V'_1) \leq_{\delta+1} (x_1, V_1^*) \leq_{\delta+1} (x_0, V'_0)$ .  $\square$

**Lemma 4.7.** *Suppose that  $\delta \geq \delta(x_0), \delta(x_1)$  and that  $x_0 \equiv_{\delta+1} x_1$ . Then for  $V_0, V_1$  open nonempty,*

$$(x_0, V_0) \leq_{\delta+1} (x_1, V_1)$$

*implies that for every  $\alpha < \omega_1$*

$$(x_0, V_0) \leq_\alpha (x_1, V_1).$$

*Proof.* By induction on  $\alpha \geq \delta + 1$ . Notice that we use the fact that  $\delta$  is higher than both  $\delta(x_0)$  and  $\delta(x_1)$ .  $\square$

We are now ready to prove the main result of this section:

**Proposition 4.8.** *If  $\delta(x_0), \delta(x_1) \leq \delta$  and  $x_0 \equiv_{\delta+1} x_1$ , then  $x_0$  and  $x_1$  are orbit equivalent.*

*Proof.* Let  $W_0$  be open and nonempty. There is  $V_0$  such that  $(x_0, V_0) \leq_{\delta+1} (x_1, W_0)$ . Using Lemma 4.6, we will be able to imitate the proof of Scott's isomorphism theorem.

We choose  $V_1 \subseteq V_0$  such that  $\overline{V_1} \subseteq V_0$  and  $\text{diam}(V_1) < 1$ . Since  $(x_0, V_0) \leq_{\delta+1} (x_1, W_0)$ , we may find  $W'_1 \subseteq W_0$  such that

$$(x_1, W'_1) \leq_{\delta+1} (x_0, V_1)$$

( here we use Lemma 4.6 ). Then choose  $W_1 \subseteq W'_1$  such that  $\overline{W_1} \subseteq W'_1$  and  $\text{diam}(W_1) < 1$ . There is  $V'_2 \subseteq V_1$  such that

$$(x_0, V'_2) \leq_{\delta+1} (x_1, W_1)$$

where again we have used Lemma 4.6. We continue in the same way: given

$$(x_0, V'_{n+1}) \leq_{\delta+1} (x_1, W_n)$$

choose  $V_{n+1} \subseteq V'_{n+1}$  such that  $\overline{V_{n+1}} \subseteq V'_{n+1}$  and  $\text{diam}(V_{n+1}) < \frac{1}{n+1}$  and we get  $W'_{n+1} \subseteq W_n$  such that

$$(x_1, W'_{n+1}) \leq_{\delta+1} (x_0, V_{n+1}).$$

We then find  $W_{n+1} \subseteq W'_{n+1}$  with  $\overline{W_{n+1}} \subseteq W'_{n+1}$  and  $\text{diam}(W_{n+1}) < \frac{1}{n+1}$ , and get  $V'_{n+2} \subseteq V_{n+1}$  with

$$(x_0, V'_{n+2}) \leq_{\delta+1} (x_1, W_{n+1}).$$

At the end of the above process we will have:

(I)

$$\cdots \subseteq V_3 \subseteq V'_3 \subseteq V_2 \subseteq V'_2 \subseteq V_1 \subseteq V_0$$

$$\cdots \subseteq W_2 \subseteq W'_2 \subseteq W_1 \subseteq W'_1 \subseteq W_0$$

such that for all  $n$

$$\begin{aligned} \text{diam}(V_n) &< \frac{1}{n}; \quad \overline{V_{n+1}} \subseteq V_n \\ \text{diam}(W_n) &< \frac{1}{n}; \quad \overline{W_{n+1}} \subseteq W_n \end{aligned}$$

(II) For all  $n \geq 1$  :

$$\begin{aligned} \overline{V_{n+1} \cdot x_0} &\subseteq \overline{V'_{n+1} \cdot x_0} \subseteq \overline{W_n \cdot x_1} \\ \overline{W_n \cdot x_1} &\subseteq \overline{W'_n \cdot x_1} \subseteq \overline{V_n \cdot x_0} \end{aligned}$$

from which we deduce

$$\cdots \subseteq \overline{V_4 \cdot x_0} \subseteq \overline{W_3 \cdot x_1} \subseteq \overline{V_3 \cdot x_0} \subseteq \overline{W_2 \cdot x_1} \subseteq \overline{V_2 \cdot x_0} \subseteq \overline{W_1 \cdot x_1}$$

By (I) we can define  $g$  as the only object of  $\bigcap V_n$  and  $h$  - the only object of  $\bigcap W_n$ . Given an  $\epsilon > 0$ , from continuity, there is an  $n \in \mathbb{N}$  such that

$$\overline{W_n \cdot x_1} \subseteq B_\epsilon(h \cdot x_1).$$

From (II)

$$\overline{V_{n+1} \cdot x_0} \subseteq \overline{W_n \cdot x_1}$$

and in particular:

$$g \cdot x_0 \in B_\epsilon(h \cdot x_1).$$

This is true for every  $\epsilon > 0$ , which is why  $g \cdot x_0 = h \cdot x_1$ . □

**Corollary 4.9.** *If  $\delta(x_0), \delta(x_1) \leq \delta$ ,  $x_0 \equiv_{\delta+1} x_1$  and  $V_0, W_0$  satisfy*

$$(x_0, V_0) \leq_{\delta+1} (x_1, W_0).$$

*Then for every  $V' \subseteq V_0$  open nonempty, there are  $g \in V'$  and  $h \in W_0$  such that  $g \cdot x_0 = h \cdot x_1$ .*

*Proof.* This is in fact what we have proved now. □

**Lemma 4.10.** *For every  $\alpha$ , the set  $\{x : \delta(x) \leq \alpha\}$  is  $\Pi_{\alpha+m(\alpha)}^0$ , for  $m(\alpha)$  natural.*

*Proof.* Reading the definition using 3.10. □

**Theorem 4.11.** *For every  $x \in X$  there is a natural number  $m$  such that  $[x] = \{y : y \equiv_{\delta(x)+m} x\}$ .*

*Proof.* Immediate from the previous Lemma and Proposition 4.8. □

## 5. HJORTH ANALYSIS AND BOREL ORBIT EQUIVALENCE RELATIONS

We will now discuss the complexity of sets of the form  $B \cdot x$  for  $B$  Borel. This discussion, apart of being interesting in its own, will be applied to the theory of Hjorth analysis.

The following is trivial:

**Proposition 5.1.** *For  $B$  Borel,  $B \cdot x$  is analytic.*

Unfortunately, we can't do better:

**Example 5.2.** (Hrushovski): Let  $2^\omega \times 2^\omega$  act on  $2^\omega$  by  $(x, y) \cdot z = x \cdot z$ . Then for any  $A \subseteq 2^\omega \times 2^\omega$ ,  $A \cdot 1$  is the projection of  $A$  on the first coordinate. Hence,  $\{F \cdot 1 : F \subseteq 2^\omega \times 2^\omega \text{ closed}\}$  is the collection of analytic subsets of  $2^\omega$ . In particular,  $B \cdot x$  for  $B$  Borel is not necessarily Borel.

**Proposition 5.3.**  *$B \cdot x$  is Borel if and only if  $B \cdot G_x$  is Borel. In particular,  $U \cdot x$  is Borel, for  $U$  open.*

*Proof.* We will give 2 different proofs:

- (1)  $B \cdot x = B \cdot G_x \cdot x$ , so we may assume that  $B$  is a collection of cosets of  $G_x$ .  $y \notin B \cdot x \iff y \notin G \cdot x \vee (\exists g \ g \cdot x = y \wedge g \cdot G_x \cap B = \emptyset) \iff y \notin G \cdot x \vee (\exists g \ g \cdot x = y \wedge g \cdot G_x \not\subseteq B) \iff y \notin G \cdot x \vee (\exists g \exists h \ g \cdot x = y \wedge h \cdot x = x \wedge g \cdot h \notin B)$ . Hence  $B \cdot x$  is co-analytic as well.
- (2) Consider  $\pi : G \rightarrow G/G_x$  the canonical projection and  $\phi : G/G_x \rightarrow G \cdot x$  the bijection between the cosets of  $G_x$  and the orbit of  $x$ .  $\phi$  is a continuous bijection between standard Borel spaces, and hence is a Borel isomorphism of the two. Hence,  $B \cdot x = \phi(\pi(B))$  is Borel if and only if  $\pi(B)$  is Borel, if and only if  $\pi^{-1}(\pi(B)) = B \cdot G_x$  is Borel. □

More can be said about the complexity of  $U \cdot x$  for  $U$  open:

**Proposition 5.4.** (1) *If  $G \cdot x$  is  $\Pi_{\alpha+1}^0$  for  $\alpha \geq 1$  then for every open  $U$ ,  $U \cdot x$  is  $\Pi_{\alpha+1}^0$ .*

(2) *If  $G \cdot x$  is  $\Sigma_\alpha^0$  then for every open  $U$ ,  $U \cdot x$  is  $\Sigma_\alpha^0$ .*

*Proof.* (1) We deal first with the case  $\alpha = 1$ , which is,  $G \cdot x$  is  $G_\delta$ . In this case, Effros' theorem is valid, and since  $U \cdot G_x$  is open in  $G \setminus G_x$ ,  $U \cdot x$  is open in  $G \cdot x$ . We deduce that  $U \cdot x$  is  $G_\delta$ .

For arbitrary  $\alpha$ , there is a sequence  $\langle B_n : n \in \omega \rangle$  of  $\Sigma_\alpha^0$  sets such that  $G \cdot x = \bigcap_{n \in \omega} B_n$ . We use Vaught transforms:

$$G \cdot x = (G \cdot x)^* = \bigcap_{n \in \omega} (B_n)^* = \bigcap_{n \in \omega} \bigcap_{m \in \omega} (B_n)^{\triangle U_m}$$

where  $U_m$  is a countable basis for the topology of  $G$ . We can then apply Theorem 2.18 and refine the topology  $\tau$  of  $X$  to a topology  $\sigma$  in which  $G \cdot x$  is  $G_\delta$ . Using the case  $\alpha = 1$ ,  $U \cdot x$  is  $G_\delta$  in  $\sigma$ , and hence  $U \cdot x$  was  $\Pi_{\alpha+1}^0$  in the original topology.

- (2) By Theorem 2.18, there is  $\tau \subseteq \sigma \subseteq \Sigma_\alpha^0(\mathbf{X}, \tau)$  Polish s.t.  $(X, \sigma)$  is a Polish  $G$ -space and  $G \cdot x$  is open in  $\sigma$ . Then  $U \cdot x$  is open in  $(G \cdot x, \sigma)$ , so it is an intersection of 2  $\Sigma_\alpha^0$  sets.

□

The first clause of the previous proposition is not true in general for  $\alpha = 0$  - one trivial example is the action of  $(\mathbb{R}, +)$  on itself.

We will now apply the above to the theory of Hjorth analysis, keeping our main goal in mind - proving that if  $E_G^X$  is Borel then Hjorth rank must be bounded.

**Proposition 5.5.**  *$y \in (V \cdot x)^{*W}$  if and only if  $(y, W) \leq_\alpha (x, V)$  for every  $\alpha < \omega_1$ . In particular, if  $(y, W) \leq_\alpha (x, V)$  for every  $\alpha$  then  $y$  and  $x$  are orbit equivalent.*

*Proof.* We first assume that  $y \in (V \cdot x)^{*W}$ . We will show that for every  $\alpha$ ,  $(y, W) \leq_{\alpha+1} (x, V)$ . So let  $W_0 \subseteq W$ . There is  $g \in W_0$  and  $h \in V$  such that

$$g \cdot y = h \cdot x.$$

We then find  $V_0 \subseteq V$  small enough around  $h$  that  $V_0 \cdot h^{-1} \cdot g \subseteq W_0$ . Hence

$$(x, V_0) \leq_\alpha (g^{-1} \cdot h \cdot x, V_0 \cdot h^{-1} \cdot g) \leq_\alpha (y, W_0)$$

where we have used Lemma 3.8.

For the other direction, by the previous proposition there is  $\alpha$  such that  $V \cdot x$  is  $\Pi_\alpha^0$ . Since  $x \in (V \cdot x)^{*V}$  is a triviality and  $(y, W) \leq_\alpha (x, V)$  is part of the assumption, Lemma 3.12 implies  $y \in (V \cdot x)^{*W}$ . □

**Proposition 5.6.** *Let  $x \in X$  and  $\alpha$  such that for every  $V \subseteq G$  open,  $V \cdot x$  is  $\Pi_\alpha^0$ . Then  $(y, W) \leq_\alpha (x, V)$  implies*

$$\forall \beta : (y, W) \leq_\beta (x, V).$$

*Proof.* Immediate using the above. □

The boundedness principle follows easily:

**Theorem 5.7.** *Let  $(G, X)$  be a Polish action and  $\mathbb{B} \subset X$  an invariant Borel set. Then  $E_G^\mathbb{B}$  is Borel if and only if there is an  $\alpha$  such that for every  $x \in \mathbb{B}$ ,  $\delta(x) \leq \alpha$ .*

*Proof.* We first assume that the rank is bounded. We use 4.11 and Sami's theorem to show that  $E_G^\mathbb{B}$  is Borel.

Now assume  $E_G^\mathbb{B}$  is Borel. There is an  $\alpha < \omega_1$  such that all orbits are  $\Pi_{\alpha+1}^0$ . Proposition 5.4 then implies that for all  $U \subseteq G$  open,  $U \cdot x$  is  $\Pi_{\alpha+1}^0$ . Using the previous proposition,  $\delta(x) \leq \alpha + 1$ . □

**Corollary 5.8.** *Let  $O \subseteq G$  be a clopen subgroup of  $G$ . Then if  $E_G^X$  is Borel then so does  $E_O^X$ . Furthermore, if Hjorth ranks of  $E_G^X$  are bounded by  $\alpha$ , then so do the Hjorth ranks of  $E_O^X$ .*

*Proof.* If  $E_G^X$  is Borel, then the complexities of  $U \cdot x$  are all less than some  $\alpha$ . The open sets of  $O$  are open sets of  $G$ , so the same  $\alpha$  bounds the complexities of  $U \cdot x$  for  $U \subseteq O$ , and in particular  $E_O^X$  is Borel. □

*Remark 5.9.* The above is not true when  $G$  is not open : there are  $H \leq G$  Polish and  $X$  a Polish  $G$  - space, such that  $E_G^X$  is Borel and  $E_H^X$  is not Borel.

The notion of Hjorth rank simplifies the proof of the following theorem of [1] and adds information about the decomposition, although some definability is lost on the way:

**Theorem 5.10.** (*Decomposition of Polish actions*) *Let  $X$  be a Polish  $G$  - Space. There is a sequence  $\{A_\zeta\}_{\zeta < \omega_1}$  of pairwise disjoint Borel subsets of  $X$  such that:*

- (1)  $A_\zeta$  is invariant, and  $\bigcup_{\zeta < \omega_1} A_\zeta = X$ . Furthermore,  $A_\zeta$  is  $\Pi_{\zeta+n(\zeta)}^0$ .
- (2)  $E_a \upharpoonright A_\zeta$  is Borel. In fact, it is  $\Pi_{\zeta+k(\zeta)}^0$ .
- (3) (Boundedness) If  $A \subseteq X$  is invariant Borel and  $E_a \upharpoonright A$  is Borel, then  $A \subseteq \bigcup_{\zeta < \alpha} A_\zeta$  for some  $\alpha < \omega_1$ .

*Proof.*  $A_\zeta$  will be the set of  $x$ 's with rank  $\zeta$ .

□

We now have all that is needed to give a positive answer to a conjecture of Hjorth:

**Theorem 5.11.** *For  $\beta$  limit, the set*

$$\mathbb{A}_\beta = \{x : [x] \text{ is } \Pi_\alpha^0 \text{ for } \alpha < \beta\}$$

*is Borel. Furthermore, it is  $\Pi_{\beta+m}^0$  for  $m \in \omega$ .*

*Proof.* We claim that this set is in fact  $\{x : \delta(x) < \beta\}$ . One direction is Theorem 4.11. The other is immediate given Proposition 5.4.

□

**Corollary 5.12.** *For  $\alpha$  limit, there are either countably many or perfectly many  $\Pi_\alpha^0$  orbits*

*Proof.* Consider the action of  $G$  on  $\mathbb{A}_\alpha$  as above. The set  $\mathbb{A}_\alpha$  is Borel invariant and the Hjorth ranks are bounded on  $\mathbb{A}_\alpha$ . Hence the orbit equivalence relation on  $\mathbb{A}_\alpha$  is Borel, and there are either countably many or perfectly many orbits in  $\mathbb{A}_\alpha$ .

□

The following is a generalization of Theorem 2.13:

**Corollary 5.13.** *The following are equivalent:*

- (1)  $E_G^{\mathbb{B}}$  is Borel.
- (2)  $\{(x, y, U, W) : U, W \in \mathfrak{B}_0; \forall \alpha < \omega_1 (x, U) \leq_\alpha (y, W)\}$  is Borel.
- (3)  $\{(x, U, W) : U, W \in \mathfrak{B}_0; \forall \alpha < \omega_1 (x, U) \leq_\alpha (x, W)\}$  is Borel.

*Proof.* (1)  $\Rightarrow$  (2) : Follows easily from the proof of Proposition 5.6.

(2)  $\Rightarrow$  (3) : Immediate.

(3)  $\Rightarrow$  (1) : Using 2.3, it is enough to show that

$$f : X \rightarrow \mathcal{F}(G)$$

$$f(x) = G_x$$

is Borel.

Hence, it suffices to show that for any  $U \subseteq G$ , the set

$$Z = \{x : \exists g \in U \ g \cdot x = x\}$$

is Borel. We claim that

$$Z = \{x : \exists V, W \in \mathfrak{B}_0 \text{ s.t. } W^{-1} \cdot V \subseteq U \text{ and } \forall \alpha (x, V) \leq_\alpha (x, W)\},$$

which is a Borel set by the assumption, so we only need to prove this claim.

Assume  $V, W \in \mathfrak{B}_0$  are such that  $W^{-1} \cdot V \subseteq U$  and  $\forall \alpha (x, V) \leq_\alpha (x, W)$ . By Proposition 5.5, there are  $g \in V$  and  $h \in W$  such that  $g \cdot x = h \cdot x$ . Hence  $h^{-1} \cdot g \cdot x = x$  so that  $x \in Z$ .

Now let  $x \in Z$ , and let  $g \in U$  such that  $g \cdot x = x$ . For every  $\alpha$  and every open set  $V$ :

$$(x, V) \leq_\alpha (x, V \cdot g^{-1}).$$

Thus, it will be enough to find  $W, V \in \mathfrak{B}_0$  such that  $V \cdot g^{-1} \subseteq W$  and  $W^{-1} \cdot V \subseteq U$ .

We will find  $W$  a neighbourhood of the identity small enough so that  $W^{-1} \cdot W \cdot g \subseteq U$ , and  $V$  a neighbourhood of  $g$  small enough so that  $V \cdot g^{-1} \subseteq W$ . □

A last characterization of Borel equivalence relations we mention is the following:

**Proposition 5.14.** *Let  $X$  be a Polish  $G$ -space.  $E_G^X$  is Borel if and only if  $R_+ = \{(x, y) : \delta(x) = \delta(y)\}$  is Borel if and only if  $R_< = \{(x, y) : \delta(x) < \delta(y)\}$  is Borel.*

*Proof.* Assume  $E_G^X$  is Borel. Then the Hjorth ranks are bounded, say by  $\gamma$ . Hence,  $\delta(x) = \delta(y)$  if and only if there is  $\alpha \leq \gamma$  such that  $\delta(x) = \alpha$  and  $\delta(y) = \alpha$ . A similar argument works for  $R_<$ .

On the other hand, assume  $E_G^X$  is not Borel. So the equivalence relation  $R_+$  has  $\aleph_1$  equivalence classes, and the well founded relation  $R_<$  is of height  $\omega_1$ . Thus  $R_<$  has uncountable height, so it cannot be analytic, let alone Borel. As for  $R_+$ , if it were Borel, there would be a perfect set of different rank elements. Extending to a  $V[G]$  such that  $V[G] \models \neg CH$ , by Shoenfield's absoluteness,  $R_+$  will still have a perfect set of different rank elements, which is a contradiction. □

In the next section we will see what can be said about complexities of rank comparisons in general.

**5.1. The Logic Action Example Revisited.** We'll consider now the logic action and see how do Hjorth and Scott analyses compare. As a corollary, we will get Theorem 2.12 of Becker and Kechris.

Let  $\mathcal{L}$  be a countable language,  $Mod(\mathcal{L})$  the Polish space of countable models of  $\mathcal{L}$ , and  $S_\infty$  "logically" acts on  $Mod(\mathcal{L})$ , as described in the introduction.

**Definition 5.15.** For  $\bar{a}, \bar{b}$  finite 1-1 same length sequences of natural numbers:

$$V_{\bar{a}, \bar{b}} = \{\sigma \in S_\infty : \sigma(\bar{a}) = \bar{b}\}.$$

The  $V_{\bar{a}, \bar{b}}$  sets form a countable basis of the topology of  $S_\infty$ .

**Lemma 5.16.** *For  $\mathcal{M}$  a countable model:  $\overline{V_{\bar{a}, \bar{b}} \cdot \mathcal{M}} = \{\mathcal{N} : Th_\Sigma(\mathcal{N}, \bar{b}) \subseteq Th_\Sigma(\mathcal{M}, \bar{a})\}$  ( $Th_\Sigma$  stands for the existential first order theory of a model).*

*Proof.*  $\mathcal{N} \in \overline{V_{\bar{a}, \bar{b}} \cdot \mathcal{M}}$  if and only if for every atomic formula  $\phi(\bar{x}, \bar{y})$ , if  $\mathcal{N} \models \phi(\bar{b}, \bar{c})$  then there is  $\sigma \in V_{\bar{a}, \bar{b}}$  such that  $\mathcal{M} \models \phi(\bar{a}, \sigma^{-1}(\bar{c}))$ . In other words, if  $\mathcal{N} \models \exists \bar{y} \phi(\bar{b}, \bar{y})$  then  $\mathcal{M} \models \exists \bar{y} \phi(\bar{a}, \bar{y})$ . □

**Proposition 5.17.** *If  $(\mathcal{M}, \bar{a}) \equiv_{\omega \cdot \alpha} (\mathcal{N}, \bar{a}')$  then for every  $\bar{b}$  finite 1-1 sequences:*



$$(\mathcal{M}, V_{\bar{a}, \bar{b}}) \leq_\alpha (\mathcal{N}, V_{\bar{a}', \bar{b}}).$$

*Proof.* By induction on  $\alpha$ . For  $\alpha = 1$ , we assume  $(\mathcal{M}, \bar{a}) \equiv_\omega (\mathcal{N}, \bar{a}')$  and want to show that

$$\overline{V_{\bar{a}, \bar{b}} \cdot \mathcal{M}} \subseteq \overline{V_{\bar{a}', \bar{b}} \cdot \mathcal{N}}$$

for any  $\bar{b}$  as above. So assume  $\mathcal{P} \in \overline{V_{\bar{a}, \bar{b}} \cdot \mathcal{M}}$ . Then by Lemma 5.16,  $Th_\Sigma(\mathcal{P}, \bar{b}) \subseteq Th_\Sigma(\mathcal{M}, \bar{a})$ . Then  $Th_\Sigma(\mathcal{P}, \bar{b}) \subseteq Th_\Sigma(\mathcal{N}, \bar{a}')$ , as they are logically equivalent, so that  $\mathcal{P} \in \overline{V_{\bar{a}', \bar{b}} \cdot \mathcal{N}}$ .

The case  $\alpha$  limit is trivial. Consider then the case  $\alpha = \beta + 1$ . We assume  $(\mathcal{M}, \bar{a}) \equiv_{\omega \cdot \beta + \omega} (\mathcal{N}, \bar{a}')$ . Let  $V_{\bar{a}\bar{c}, \bar{b}\bar{d}} \subseteq V_{\bar{a}, \bar{b}}$ . It will be enough to find  $\bar{c}'$  such that

$$(\mathcal{N}, V_{\bar{a}'\bar{c}', \bar{b}\bar{d}}) \leq_\beta (\mathcal{M}, V_{\bar{a}\bar{c}, \bar{b}\bar{d}}).$$

By the definition of  $\equiv_{\omega \cdot \beta + \omega}$ , there is  $\bar{c}'$  such that  $(\mathcal{N}, \bar{a}'\bar{c}') \equiv_{\omega \cdot \beta} (\mathcal{M}, \bar{a}\bar{c})$ . Then by the induction hypothesis we get the above.  $\square$

**Corollary 5.18.** *Let  $\mathbb{B} \subseteq Mod(\mathcal{L})$  be an invariant Borel subset. Let  $\alpha < \omega_1$  be such that for every  $\mathcal{M} \in \mathbb{B}$ , the Hjorth rank  $\delta_H(\mathcal{M})$  is bounded by  $\alpha$ . Then the Scott rank  $\delta_S(\mathcal{M})$  is bounded by  $\omega \cdot (\alpha + 1)$ .*

*Proof.* Let  $\mathcal{M} \in \mathbb{B}$ , and assume  $(\mathcal{M}, \bar{a}) \equiv_{\omega \cdot (\alpha + 1)} (\mathcal{M}, \bar{b})$ . Then by the previous proposition:

$$(\mathcal{M}, V_{\bar{a}, \bar{c}}) \leq_{\alpha + 1} (\mathcal{M}, V_{\bar{b}, \bar{c}}).$$

Then by the assumption, for every  $\gamma < \omega_1$ ,  $(\mathcal{M}, V_{\bar{a}, \bar{c}}) \leq_\gamma (\mathcal{M}, V_{\bar{b}, \bar{c}})$ , which is why there are  $g \in V_{\bar{a}, \bar{c}}$  and  $h \in V_{\bar{b}, \bar{c}}$  such that  $g \cdot \mathcal{M} = h \cdot \mathcal{M}$ . The permutation  $h^{-1} \cdot g \in V_{\bar{a}, \bar{b}}$  shows that  $(\mathcal{M}, \bar{a}) \simeq (\mathcal{M}, \bar{b})$ .  $\square$

**Corollary 5.19.** *(Becker - Kechris) Let  $\mathbb{B} \subseteq Mod(\mathcal{L})$  be an invariant Borel subset. If  $E_G^\mathbb{B}$  is Borel then there is an  $\alpha < \omega_1$  such that the Scott ranks of the elements of  $\mathbb{B}$  are all bounded by  $\alpha$ .*

## 6. HJORTH RANK AND COMPUTABLE ORDINALS

Nadel [12] has shown that for the logic action, the Scott rank of a model  $M$  is at most  $\omega_1^{ck(M)}$ . The model  $M$  is identified with a sequence of 0's and 1's the moment we begin to talk about  $Mod(L)$  as a topological space, so the meaning of  $\omega_1^{ck(M)}$  is clear.

In the general Polish action case,  $\omega_1^{ck(x)}$  has no obvious meaning a priori. However, we would like it to be the first ordinal not computable in an oracle that knows all about the action of  $G$  on that specific  $x \in X$ . The following definition follows:

**Definition 6.1.** Fix a basis  $\mathfrak{B}_0 = \langle V_k : k < \omega \rangle$  for the topology of  $G$  and a basis  $\langle U_l : l < \omega \rangle$  for the topology of  $X$ . For  $x \in X$ , define  $x_G : \omega \rightarrow 2$  as follows:

$$x_G(\langle k, l \rangle) = 1 \iff (V_k \cdot x) \cap U_l \neq \emptyset.$$

$x_G$  codes the action of  $G$  on  $X$ . The map  $x \rightarrow x_G$  is a Borel map from  $X$  to  $2^\omega$ . We'll usually abuse notation and write  $x$  instead of  $x_G$ . In particular,  $\omega_1^{ck(x)}$  stands for  $\omega_1^{ck(x_G)}$ .

We show Nadel's theorem for Hjorth analysis:

**Theorem 6.2.** *For every  $x \in X$ ,  $\delta(x) \leq \omega_1^{ck(x)}$ . In particular, the orbit of  $x$  is  $\Pi_{\omega_1^{[x]} + \mathbf{n}}^0$ , for some  $n \in \omega$ , where  $\omega_1^{[x]} = \min\{\omega_1^y : yEx\}$ .*

For the proof, we first analyze the lightface complexity of  $\leq_\alpha$ . First some ad-hoc definitions:

**Definition 6.3.** We say that  $k^*$  is **contained** in  $k$  if  $V_{k^*} \subseteq V_k$ . We say that  $k^*, m^* \in \omega$  are **fine** with respect to  $k, m \in \omega$  if  $\overline{V_{k^*}} \subseteq V_k$  and  $\overline{V_{m^*}} \subseteq V_m$ . We assume that both these relations are recursive.

We define the relation  $S$  as follows:

For  $x_G \in 2^\omega$  as above,  $z \in 2^\omega$ ,  $R \in LO$ :  $(x_G, z, R) \in S \iff$

$$(\forall k, m : z_{\langle n, k, m \rangle} = 1 \iff (x, V_k) \leq_{tp(n)} \text{ w.r.t. } R (x, V_m))$$

$S$  says that  $z$  codes all the information about  $\leq_\alpha$  on  $x$  for  $\alpha$ 's smaller than  $tp(R)$ . If  $R$  is not a well order, we don't care what  $S$  means.

**Lemma 6.4.**  $S$  is arithmetic.

*Proof.* Easy.

*Proof.* (of Theorem 6.2): Denote  $\omega_1^{ck(x)}$  by  $\epsilon$ . Assume by way of contradiction that  $\delta(x) > \epsilon$ . Then there are  $k, m$  and  $k^*, m^*$  fine with respect to  $k, m$  such that:

$$(x, V_k) \leq_\epsilon (x, V_m)$$

but

$$(x, V_{k^*}) \not\leq_{\epsilon+1} (x, V_{m^*})$$

so there is  $i$  contained in  $k^*$  such that for every  $j$  contained in  $m^*$ :

$$(x, V_j) \not\leq_\epsilon (x, V_i)$$

For each  $j$  as above, we'll choose  $\alpha_j < \epsilon$  such that  $(x, V_j) \leq_{\alpha_j} (x, V_i)$  but  $(x, V_j) \not\leq_{\alpha_j+1} (x, V_i)$ .

*Claim 6.5.*  $\alpha_j$  are cofinal in  $\epsilon$ .

*Proof.* Otherwise, there is  $\beta < \epsilon$  such that all  $\alpha_j$ 's are less than  $\beta$ . But  $(x, V_{k^*}) \leq_\epsilon (x, V_{m^*})$  so there is  $j$  contained in  $m^*$  such that  $(x, V_j) \leq_{\beta+1} (x, V_i)$ . Which is a contradiction. □

□

However:

*Claim 6.6.* The set  $W = \{(j, R) : tp(R) = \alpha_j; j \text{ is contained in } m^*\}$  is  $\Sigma_1^1(x)$ .

*Proof.*  $(j, R) \in W \iff$  all of the following are satisfied:

- (1)  $R$  is a linear order computable by  $x$ .  $j$  is contained in  $m^*$ .
- (2) There exist a program  $e$ , natural numbers  $n, l$  and a sequence  $z$  such that  $[e](x)$  is  $tp(R) + 2$ ,  $n$  is the last element of  $[e](x)$ ,  $l$  is its immediate predecessor,  $(x, z, [e](x)) \in S$ ,  $z_{\langle l, j, i \rangle} = 1$  and  $z_{\langle n, j, i \rangle} = 0$ .
- (3) For every program  $e$ , if  $[e](x)$  is a well order, **then** there is a program  $e'$ , a sequence  $z$ , and a natural number  $n$  such that  $[e'](x)$  is  $[e](x) + 1$ ,  $n$  is the last element of  $[e'](x)$ ,  $(x, z, [e'](x)) \in S$ , and if  $z_{\langle n, j, i \rangle} = 0$  then there is an embedding  $i : (\omega, R) \rightarrow (\omega, [e](x))$ , order preserving and onto a proper initial segment of  $[e](x)$ .

□

The last 2 claims contradict the boundedness theorem. □

We next generalize Sacks' theorem:

**Theorem 6.7.** *If for every  $x$ ,  $\delta(x) \leq \alpha$  or  $\delta(x) < \omega_1^{ck(x)}$ , then Hjorth ranks are bounded.*

First a useful lemma:

**Lemma 6.8.** *There are  $\Sigma_1^1$  and  $\Pi_1^1$  formulas  $\phi(x, R)$  and  $\psi(x, R)$  such that if  $R \in WO$  then  $\phi(x_G, R)$  if and only if  $\psi(x_G, R)$  if and only if*

$$\forall k \forall m (\forall k^* \forall m^* \text{ fine w.r.t. } k, m) \quad (x, V_k) \leq_{tp(R)} (x, V_m) \Rightarrow (x, V_{k^*}) \leq_{tp(R)+1} (x, V_{m^*})$$

*if and only if  $\delta(x) \leq tp(R)$ .*

*Proof.* Consider the following  $\Sigma_1^1$  formula:

There exist a program  $e$ , natural numbers  $n, l$  and a sequence  $z$  such that  $[e](x)$  is  $tp(R) + 2$ ,  $(x, z, [e](x)) \in S$ ,  $n$  is the last element in  $[e](x)$ ,  $l$  is its immediate predecessor, and for all  $k, m$  and for all  $k^*, m^*$  fine with respect to  $k, m$ , if  $z_{\langle l, k, m \rangle} = 1$  then  $z_{\langle n, k^*, m^* \rangle} = 1$ .

Under the above conditions, this can also be written by a  $\Pi_1^1$  formula. □

*Proof. (of Theorem 6.7)* Fix  $y$  that computes  $\alpha$ . So for every  $x$ ,  $\delta(x) < \omega_1^{ck(x, y)}$ . Denote by  $H$  the set

$$\{(R, x) : tp(R) = \delta(x)\}$$

We claim that  $H$  is  $\Sigma_1^1(y)$ :

$(R, x) \in H$  if and only if all of the following are satisfied:

- (1)  $R$  is a linear order computable by  $x, y$ .
- (2)  $\delta(x) \leq tp(R)$ .
- (3) For every program  $e$ , if  $e[x, y]$  computes a well order  $Q$  and  $\delta(x) \leq tp(Q)$ , then there is  $i : (\omega, R) \rightarrow (\omega, Q)$ , order preserving and onto an initial segment of  $Q$ .

Using Lemma 6.8, 2 and 3 are  $\Sigma_1^1(y)$ .

By the boundedness theorem, the Hjorth ranks are bounded. □

We apply the above to compute the complexity of rank comparisons:

**Corollary 6.9.**  $\{(x, y) : \delta(x) = \delta(y)\}$  is analytic.  $\{(x, y) : \delta(x) < \delta(y)\}$  is  $\Pi_1^1$ .  $\{(x, y) : \delta(x) \leq \delta(y)\}$  is analytic.

*In light of Proposition 5.14, these are optimal.*

*Proof.* We remind that  $x \rightarrow x_G$  is Borel.

$\delta(x) = \delta(y)$  if and only if for every program  $e$ , if  $e[x_G, y_G]$  computes a well order  $R$ , then  $\delta(x) \leq tp(R) \iff \delta(y) \leq tp(R)$ . We now use Lemma 6.8.

$\delta(x) < \delta(y)$  if and only if there is a program  $e$  such that  $e[y_G]$  computes a well order  $R$ , and  $\delta(x) \leq tp(R)$  but  $\delta(y) > tp(R)$ . Again, we use Lemma 6.8. □

Which leads us to the following:

**Corollary 6.10.** *Let  $X$  be a perfect Polish  $G$  - space,  $A_\alpha = \{x : \delta(x) = \alpha\}$ . There is an  $\alpha < \omega_1$  such that  $A_\alpha$  is non meager.*

**Fact 6.11.** (Kechris [7]) *Let  $X$  be a Polish space,  $A \subseteq X$ ,  $\langle A_\alpha : \alpha \in \omega_1 \rangle$  a partition of  $A$  into a family of disjoint meager sets in  $X$ . Let  $\leq^*$  be defined by:  $x \leq^* y$  if both  $x, y$  are in  $A$ , and the  $\alpha$  such that  $x \in A_\alpha$  is smaller or equal than the  $\beta$  such that  $y \in A_\beta$ . Then if  $\leq^*$  has the BP, then  $A$  is meager.*

*Proof. (of the Corollary)* Using the fact and Corollary 6.9. □

**Corollary 6.12.** *Let  $X$  be a Polish  $G$  - space which is a counterexample to Vaught conjecture. Let  $Y \subseteq X$  be any non meager set. Then  $Y$  has a non meager orbit. Hence, the union of all meager orbits must be meager.*

*Proof.*  $Y = \bigcup_{\alpha < \omega_1} (Y \cap A_\alpha)$ . Using Fact 6.11 again, one of the  $Y \cap A_\alpha$  is non meager. But it has only countably many orbits, so one of them is non meager. □

**Corollary 6.13.** *Let  $X$  be a counterexample to Vaught. Then there is a non empty and at most countable collection  $\langle C_i : i \in \omega \rangle$  of  $G_\delta$  orbits, some of them non meager, such that  $C = \bigcup_{i \in \omega} C_i$  is comeager. At least one of those orbits is not  $F_\sigma$ .*

*Proof.* By the previous corollary, there are non meager orbits. All of them must be  $G_\delta$  (by Theorem 2.1) and there are at most countably many  $G_\delta$  orbits. Fix  $C_i$  an enumeration of the  $G_\delta$  orbits (which contains all the non meager ones). By the previous corollary,  $C$  is comeager.

Now, if all these orbits are  $F_\sigma$ , then  $C$  is  $F_\sigma$ . Hence we can consider the action of  $G$  on the Polish space  $X - C$ . It will also be a counterexample to Vaught, so it must have a non meager orbit. But a non meager orbit is  $G_\delta$ , which is a contradiction. □

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